Pricing Callable Bonds Based on Monte Carlo Simulation Techniques

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ABSTRACT
In this paper, a Monte Carlo method, which is based on some new simulation techniques proposed recently, is presented to numerically price the callable bond with several call dates and notice under the Cox-Ingersoll-Ross (CIR) interest rate model. The corresponding algorithms are also presented to practical callable bond pricing. The numerical experiments show that this method works very well for callable bond under the CIR interest rate model.

Keywords: Callable Bond; Monte Carlo Simulation; CIR Model; Embedded Option Pricing

1. Introduction
A callable bond is a bond that allows the issuer to buy back the bonds from the bond holders at pre-specified prices on the pre-specified call dates. Therefore, a callable bond is a straight bond embedded with a call of European option (a single call date) or Bermudan option (several call dates). However, this option is an integral part of a bond, and cannot be traded alone, and hence, its prices cannot be observed. Thus, the callable bond pricing must be involved in the pricing problem of the corresponding option.

There are some different approaches for pricing callable bonds. The first approach is based on the Black-Derman-Toy model, which was presented in [1] (2006), with the discrete simulation of binary tree. With the help of the risk-neutral valuation, the second approach is to obtain a partial differential equation (PDE) subject to appropriate boundary conditions based on the equilibrium interest rate model. Since it is very difficult to analytically solve this PDE, some different discretizations and different numerical methods have been proposed. Büttler in [2] (1995) applied finite difference method to find the evaluation of callable bonds. Büttler and Waldvogel in [3] (1996) derived an analytic expression for the Green's function of the corresponding PDE for certain specific interest rate models, and developed a semi-analytic method for pricing callable bonds with notice. As the further development, the finite volume method was used by D'Halluin et al. in [4] (2001), and the finite element method was considered by Farto and Vázquez in [5] (2005) for the numerically pricing callable bonds with notice. Recently, a dynamic programming approach was proposed by Ben-Ameur et al. in [6] (2007) for numerically pricing options embedded in bonds. In this dynamic programming approach they used finite difference method and solved the Green’s function by conditional distributions and expectations with piecewise-linear approximation.

Meanwhile, in the last decade, many new numerical schemes for simulations of interest rate models, especially, the Cox-Ingersoll-Ross (CIR) interest rate model, have been proposed. For instance, the balanced implicit method (BIM) was proposed by Milstein et al. in [7] (1998), the balanced Milstein method (BMM) was developed by Kahl and Schurz in [8] (2006). Also, the exact transition distribution method (ETD) is considered to simulate the square-root diffusions (e.g. see [9]). Recently, a new splitting-step scheme was presented by Ding and Chao in [10] (2009). In this paper, based on these new simulation techniques we present a Monte Carlo method to numerically price the Bermudan-type callable bond with notice.

This paper is structured as follows. After this introduction, the interest rate models are reviewed, and several numerical simulation techniques are surveyed in Section 2. Then, based on these simulation techniques, an efficient Monte Carlo method is presented to price the callable bond with several call dates and notice under the CIR interest rate model in Section 3. The corresponding algorithms are presented in this section. Finally, numerical experiments for a practical callable bond with 10 call dates and 2 months notice are provided in Section 4. The
numerical results of these experiments are also presented in this section, as well as some useful conclusions.

2. Simulations of Interest Rate Models

Pricing financial derivatives depends on the description of the dynamic process of underlying assets. Since the underlying asset of callble bond is the interest rate, we focus on the mathematical models for the interest rate. These models can be divided as single factor models and multiple factor models by the number of status variables.

The first well-known single factor model was proposed by Vasicek in [11] (1977). In this model, the interest rate \( r(t) \) is given by the stochastic differential equation (SDE):

\[
dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t),
\]

where \( \kappa, \theta \) and \( \sigma \) are all strictly positive constants, and \( W(t) \) is a standard Brownian motion. In detail, \( \kappa \) represents the speed at which \( r(t) \) reverts back to the long-term mean \( \theta \), while \( \sigma \) is the local volatility of short-term interest rates. The Ornstein-Uhlenbeck process is employed in this model for its key feature as the mean-reverting structure.

The Vasicek’s model has two significant failings. First, the interest rate can become negative; Second, empirical evidence suggests that the volatility of \( r(t) \) is not constant as the interest rate can become negative; hence, this model does not accurately describe the behavior of the interest rate.

In the last decade, many efficient new numerical schemes have been proposed for the CIR model (1) with positivity preservation. In the following, we survey these schemes, which are employed in the numerical experiments in Section 4.

Let \( T > 0 \) and let \( N \) be a positive integer. In the following we denote \( \Delta = T/N \), and set \( t_0 = 0 \) and \( t_k = k\Delta \) for each \( k = 1, \ldots, N \), i.e. \( t_0 < t_1 < \cdots < t_N \) is a partition of \([0,T]\). We also denote \( r_k = r(t_k) \) for each \( k \). We assume that \( \xi_1, \ldots, \xi_N \) are \( N \) independent random variables having a common standard normal distribution.

The balanced implicit method (BIM) was proposed by Milstein et al. in [7]. The discretisation of the CIR model (1) by the BIM is given by

\[
r_k = r_{k-1} + \kappa(\theta - r_{k-1})\Delta + \sigma\sqrt{r_{k-1}} \Delta \cdot \xi_k + \varphi(r_{k-1})(r_{k-1} - r_k),
\]

for each \( k = 1, \ldots, N \), where \( \varphi(x) \) is called the control function, and here it is given by

\[
\varphi(r_{k-1}) = \kappa \Delta + \sqrt{\Delta} \cdot \xi_k \cdot C
\]

with \( C = a/\sqrt{r_{k-1}} \) for \( r_{k-1} > \epsilon \) and \( C = a/\sqrt{\epsilon} \) elsewhere. Here \( \epsilon \) is selected as small as possible but halting the computation.

The balanced Milstein method (BMM) was developed by Kahl and Schurz in [8]. For CIR model (1) the BMM scheme is given by

\[
r_k = r_{k-1} + \kappa \Delta (\theta - r_{k-1}) + \sigma \sqrt{\Delta} \cdot \xi_k + \frac{1}{4} \sigma^2 \Delta (\xi_k^2 - 1) + \kappa\Delta (r_{k-1} - r_k),
\]

for each \( k = 1, \ldots, N \). BIM and BMM schemes preserve positivity for the CIR model (1) as \( \Delta \) tends to zero.

In the book [9], an algorithm for simulation of the CIR model (1) by the exact transition distribution (ETD) method is given in the following.

The CIR model (1) with \( d = 4\theta\kappa/\sigma^2 \):

Case 1: \( d > 1 \)

for \( k = 0, \ldots, N-1 \)

\[
c \leftarrow \sigma^2 \left(1 - e^{-\sigma(t_k)}\right)/(4\kappa)
\]

\[
\lambda \leftarrow r_k e^{-\sigma(t_k)} / c
\]

generate \( Z \sim N(0,1) \)

generate \( X \sim \chi^2_{d-1} \)

\[
r_{k+1} \leftarrow c \left[ (Z + \sqrt{\lambda})^2 + X \right]
\]

end

Case 2: \( d \leq 1 \)

for \( k = 0, \ldots, N-1 \)

\[
c \leftarrow \sigma^2 \left(1 - e^{-\sigma(t_k)}\right)/(4\kappa)
\]

\[
\lambda \leftarrow r_k e^{-\sigma(t_k)} / c
\]

generate \( N \sim \text{Positon}(\lambda/2) \)

generate \( X \sim \chi^2_{d+2N} \)

\[
r_{k+1} \leftarrow cX
\]

end
The advantage of this algorithm is the strict positivity preservation, comparing the conditional preservations of the two methods above. However, the ETD method suffers so great cost of computational time, and it also seems to be relatively unsuitable in our numerical experiments.

Recently, an efficient splitting-step scheme for the CIR model (1) was proposed by Ding and Chao in [10]. This new scheme, which is called the DC scheme here, is given as

\[ r_k = e^{x_k \left( \sqrt{\frac{\kappa}{2}} + \frac{1}{2} \sigma \sqrt{\frac{\kappa}{2}} \right)^2 + \frac{1}{\kappa} \left( \kappa \theta - \frac{1}{4} \sigma^2 \right) \left( 1 - e^{x_k} \right)} \]  

(5)

for each \( k = 1, \ldots, N \). This scheme preserves the positivity for the simulation in the case: \( 4k \theta > \sigma^2 \), and it takes less computational time in comparison to BIM and BMM schemes.

3. The Monte Carlo Method

We consider a Bermudan callable bond which has \( m+n \) pre-specified coupon dates:

\[ t_0 < t_1 < \cdots < t_m < t_{J-n} < \cdots < t_{J-1} < t_J, \]

where the bond may be redeemed at the last \( n \) dates (call dates): \( t_{J-n}, \ldots, t_{J-1} \). As in Figure 1, the notice period between each notice date and corresponding call date is denoted as \( \tau \). For convenience, \( t_{J-}\tau = t_{J-1} - \tau \) is denoted the \((n-j+1)th\) notice date for each \( j = n, \ldots, 1 \). In general, the time period between two coupon dates is one year, and the annual coupon payment is denoted as \( \eta \). At the call date \( t_{J-1} \), the call (or strike) price is defined as \( X_{J-1} \), the call (or strike) price.

Let \( E[\cdot] \) and \( P(\cdot) \) denote the conditional expectation and conditional probability under the risk-neutral probability measure \( P \). For two dates \( t_0 \leq t_i \leq t_j \), we define the discount factor over the time period \( [t_i, t_j] \) when \( r(t) = \rho \):

\[ E[\rho; i, j] = E\left[ \exp\left(-\int_{t_i}^{t_j} r(s) \, ds\right) \right] | r(t_i) = \rho \].

where \( r(t) \) is the instantaneous interest rate with the initial value \( r(t_0) = \rho_0 \). For two notice dates \( t_{J-1} \leq t_{J-j} \), we also denote:

\[ P(\rho; (J-j), (J-i)) = P\left( r(t_{J-j}) \geq \rho^*_j, r(t_{J-i}) = \rho \right), \]

Figure 1. The call dates and the corresponding notice dates of Bermuda callable bond.
with $\gamma = \sqrt{\zeta^2 + 2\sigma^2}$ and the sum of the risk premium $\zeta$, which is a parameter. Also, we can approximate the root $\rho_{j-1}^*$ by computing uncall and call values for the different values of $\rho$ via the Monte Carlo simulation.

Now, the value of the bond at the notice date $t_{j-1}$ is given by

$$K_1 \left[ \rho; (J - 1)^{j}; J \right] = K_1 \left[ \rho; (J - 1)^{j}; J \right]_{\text{uncall}} \cdot 1(\rho \geq \rho_{j-1}^*)$$

$$+ K_1 \left[ \rho; (J - 1)^{j}; J \right]_{\text{call}} \cdot 1(\rho < \rho_{j-1}^*)$$

where $1(\rho \geq \rho_{j-1}^*)$ and $1(\rho < \rho_{j-1}^*)$ are indicators of sets $\{ \rho; \rho \geq \rho_{j-1}^* \}$ and $\{ \rho; \rho < \rho_{j-1}^* \}$ respectively.

We then consider the bond value at the second notice date $t_{j-2}$. Under the condition: $r(t_{j-2}) = \rho$, if the bond is uncalled, its value is given by

$$K_2 \left[ \rho; (J - 2)^{j}; J \right]_{\text{uncall}} = E \left[ K_1 \left[ r(t_{j-2}) ; (J - 1)^{j}; J \right] \right]$$

$$\times \exp \left[ -\int_{t_{j-2}}^{t_{j-1}} r(s) \, ds \right] r(t_{j-2}) = \rho$$

$$+ \eta \cdot E \left[ \rho; (J - 2)^{j}; J - 2 \right]_{\text{call}}$$

Combining the expression (9) we have

$$K_2 \left[ \rho; (J - 2)^{j}; J \right]_{\text{uncall}} = \left[ 1 + \eta \right] E \left[ \rho; (J - 2)^{j}; J \right]$$

$$+ \eta \left[ X_{j-2} + \eta \right] E \left[ \rho; (J - 2)^{j}; J - 1 \right]$$

$$\times \left[ 1 - P \left( \rho; (J - 2), (J - 1) \right) \right]$$

$$+ \eta E \left[ \rho; (J - 2)^{j}; J - 2 \right]$$

On the other hand, if the bond is called, under the given condition: $r(t_{j-2}) = \rho$, the value is given by

$$K_2 \left[ \rho; (J - 2)^{j}; J \right]_{\text{call}} = \left( X_{j-2} + \eta \right) E \left[ \rho; (J - 2)^{j}; J - 2 \right]$$

Therefore, the bond value at the second notice date $t_{j-2}$ is given by

$$K_2 \left[ \rho; (J - 2)^{j}; J \right] = K_2 \left[ \rho; (J - 2)^{j}; J \right]_{\text{uncall}} \cdot 1(\rho \geq \rho_{j-1}^*)$$

$$+ K_2 \left[ \rho; (J - 2)^{j}; J \right]_{\text{call}} \cdot 1(\rho < \rho_{j-1}^*)$$

where $\rho_{j-2}^*$ is the break-even interest rate at the second notice date $t_{j-2}$, which can be found as the first break-even interest rate $\rho_{j-1}^*$.

Continuously, we can obtain the values of callable bond at the $j$th notice date $t_{j}$ as

$$K_j \left[ \rho; (J - j)^{j}; J \right] = K_j \left[ \rho; (J - j)^{j}; J \right]_{\text{uncall}} \cdot 1(\rho \geq \rho_{j-1}^*)$$

$$+ K_j \left[ \rho; (J - j)^{j}; J \right]_{\text{call}} \cdot 1(\rho < \rho_{j-1}^*)$$

where $\rho_{j-1}^*$ is the break-even interest rate at the $j$th notice date $t_{j-1}$.

Consequently, we get the price of the callable bond at the present date $t_0 = 0$ with the initial interest rate $r(t_0) = r_0$:

$$K \left[ \rho; 0; J \right] = E \left[ K_k \left[ r(t_{j-1}); (J - n)^{j}; J \right] \right] r(t_0) = r_0$$

$$\times E \left[ \rho; 0; (J - n)^{j} \right] \eta \sum_{i=1}^{n} E \left[ \rho; 0; i \right]$$

Now, by applying the simulation technique to the interest rate $r(t)$ and using the Monte Carlo method to approximate the corresponding integrals $E[\rho; i, j]$ and the corresponding probabilities $P(\rho; (J - j)), (J - i)$, we can obtain a numerical approximation of the price $K \left[ \rho; 0; J \right]$.

4. Numerical Experiments

In this section, we do numerical experiments via our methods to price a callable bond issued by the Swiss Confederation with an annual coupon of 4.25%. Here $t_0$ is December 23, 1991, and $t_j$ is December 31, 2012. The protection period is 10 years until year 2002. The notice period is two months. And the call prices are $X_{j-1} = \cdots = X_{j-5} = 1$, $X_{j-6} = 1.005$, $X_{j-7} = 1.01$, $X_{j-8} = 1.015$, $X_{j-9} = 1.02$ and $X_{j-10} = 1.025$, respectively.

From [3], the model parameters for the CIR model are $\kappa = 0.54958046$, $\theta = 0.38757496$, $\sigma = 0.034846851$. The initial interest rate $r_0$ is 0.0752280589, and the price of straight bond is 0.8114. The break-even interest rates are $\rho_{j-1}^* = 0.0338871564$, $\rho_{j-2}^* = 0.0179273733$, $\rho_{j-3}^* = 0.00997892562$, $\rho_{j-4}^* = 0.0048817260$, $\rho_{j-5}^* = 0.0015784739$ and $\rho_{j-6}^* = \cdots = \rho_{j-10}^* = 0$, which are given in [3]. Although the break-even interest rates can be obtained via our methods by Equation (6), the results are lack of precision. Therefore we use the results from [3] directly and these break-even interest rates are computed by Equation (7).
Table 1. Numerical results for four methods\(^a\).

<table>
<thead>
<tr>
<th>Nb</th>
<th>BIM</th>
<th>BMM</th>
<th>DC</th>
<th>ETD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Callable bond</td>
<td>0.8814</td>
<td>0.7967</td>
<td>0.8089</td>
<td>0.8575</td>
</tr>
<tr>
<td>Call option</td>
<td>–0.07</td>
<td>0.0147</td>
<td>0.0025</td>
<td>–0.0461</td>
</tr>
<tr>
<td>Error</td>
<td>8.33E–2</td>
<td>1.40E–3</td>
<td>1.08E–2</td>
<td>5.94E–2</td>
</tr>
</tbody>
</table>

Table 2. Numerical results for different \(N\)s via BMM method\(^b\).

<table>
<thead>
<tr>
<th>Nb</th>
<th>240</th>
<th>480</th>
<th>960</th>
</tr>
</thead>
<tbody>
<tr>
<td>Callable bond</td>
<td>0.7967</td>
<td>0.8009</td>
<td>0.7974</td>
</tr>
<tr>
<td>Call option</td>
<td>0.0147</td>
<td>0.0105</td>
<td>0.014</td>
</tr>
<tr>
<td>Error</td>
<td>1.40E–03</td>
<td>2.80E–03</td>
<td>7.00E–04</td>
</tr>
</tbody>
</table>

Table 3. Numerical results for different \(N\)s via DC method\(^c\).

<table>
<thead>
<tr>
<th>Nb</th>
<th>240</th>
<th>480</th>
<th>960</th>
</tr>
</thead>
<tbody>
<tr>
<td>Callable bond</td>
<td>0.8089</td>
<td>0.8058</td>
<td>0.7976</td>
</tr>
<tr>
<td>Call option</td>
<td>0.0025</td>
<td>0.0055</td>
<td>0.0138</td>
</tr>
<tr>
<td>Error</td>
<td>1.08E–02</td>
<td>7.70E–03</td>
<td>5.00E–04</td>
</tr>
</tbody>
</table>

\(a\)All prices of callable bond are computed by the average over 50,000 simulating paths.

\(b\)\(N\) is the number of time-discretized points in the simulation of interest rate.

\(c\)All figures for the callable bond are rounded to four significant digits from the 15-digit results.

\(d\)All prices of the embedded call option all per face value.

\(e\)Error is the absolute difference between callable bond price and 0.7981, which is given in [3].

Tables 1-3 give the price of this callable bond via different simulation methods. All results in the numerical experiments show that BMM and DC schemes are more efficient than others. And the Monte Carlo method works very well for pricing callable bonds.

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